ORBITS IN THE ANTI-INVARIANT SUBLATTICE OF THE K3-LATTICE

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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When a K3-surface X doubly-covers an Enriques surface, the covering transformation induces an involution on $H^2(X,\mathbb{Z})$. This cohomology group forms a lattice L_X under the cup-product, and as such is isometric to $E_8^2 \oplus U^3 =: \Lambda$. Its anti-invariant sublattice is denoted by L_X^- and it is isometric to $E_8(2) \oplus U(2) \oplus U =: \Lambda^-$. In this thesis, we will determine the number of orbits of primitive cohomology classes in Λ^- under the action of its self-isometries. We will also derive some conclusions on certain divisors of the moduli space of Enriques surfaces. Also a short survey on finiteness results of linear system of curves on K3 and Enriques surfaces is given. Some of the new results in this thesis also appear in [9].

Keywords: Lattices, K3 surfaces, Enriques surfaces.

ÖZET

K3 ÖRGÜSÜNÜN TERS-DEĞİŞMEZ ALTÖRGÜSÜNDEKİ YÖRÜNGELER

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Bir X K3 yüzeyi Enriques yüzeyini örttüğünde örtü dönüşümü X yüzeyinin ikinci kohomolojisinde bir dürme tanımlar. Bu grup L_X ile gösterilir, topolojik kesişim indeksiyle beraber bir örgü yapısına sahiptir ve bu haliyle $E_8^2 \oplus U^3 =: \Lambda$ örgüsüne eşölçevli olur. Bunun ters-değişmez altörgüsü L_X^- ile gösterilir ve $E_8(2) \oplus U(2) \oplus U =: \Lambda^-$ örgüsüne eşölçevlidir. Bu tezde Λ^- içindeki ilkel kohomoloji sınıflarının yörünge sayısını tespit ettik. Bunun yanında Enriques yüzeylerinin örnek uzaylarındaki bölenler üzerine birtakım sonuçlar elde ettik. Ayrıca K3 ve Enriques yüzeyleri üzerindeki eğrilerin doğrusal sistemlerinin sonluluğu hakkında bilinen bazı teoremlerin kısa bir özetini sunduk. Bu tezdeki yeni sonuçların bir kısmı [9] numaralı makalede de yer almıştır.

Anahtar sözcükler: Örgüler, K3 yüzeyi, Enriques yüzeyi.

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Ankara

Contents

Chapter 1

Introduction

In this chapter we give basic definitions and facts from the theory of algebraic surfaces and integral lattices.

1.1 K3 and Enriques surfaces

A compact complex surface X is called a *K3 surface* if the irregularity $q(X) = 0$ and the canonical line bundle $\mathcal{K}_X$ is trivial (*i.e.* $= \mathcal{O}_X$).

Since $q = 0$, we have $b_1 = \dim H^1(X, \mathbb{Z})_0 = 0$, because by Hodge Decomposition $q = 2\,b_1$. Moreover, $H^1(X, \mathbb{Z})$ has no torsion (see [2, §VIII, 3.2]). Therefore $H_1(X, \mathbb{Z})$ and hence $\pi_1(X)$ are trivial, that is X is simply-connected (see [6, §2.A]).

Since the canonical bundle $\mathcal{K}_X$ is trivial, the geometric genus $p_g(X) = \dim H^{0,2}(X) = \dim H^2(X, \Omega^0_X) = \dim H^2(X, \mathcal{O}_X) = \dim H^0(X, \mathcal{K}_X) = \dim H^0(X, \mathcal{O}_X) = 1$, by Serre Duality [3, §I.11]. Thus, there is a nowhere-vanishing holomorphic 2-form Ω on X.

We have $\chi(\mathcal{O}_x) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 1 - q + p_g = 2$, which implies by Noether formula (see [3, §I.14]) that $\chi(X) = 12\,\chi(\mathcal{O}) - c_1(K_X)^2 = 24$

(here we use that the first Chern class map c_1 is injective for K3 surfaces, because $H^1(X, \mathcal{O}_X) = 0$ since $q = 0$ and it has no torsion). Since $b_0 = b_4 = 1$, $b_1 = b_3 = 0$ by Poincaré duality, this implies that $b_2 = \dim H^2(X, \mathbb{C}) = 22$. So we have $\operatorname{rank} H^2(X, \mathbb{Z}) = 22$.

The fact that $H^1(X, \mathbb{Z}) = 0$ implies that $H^2(X, \mathbb{Z})$ is torsion-free: Indeed, any compact manifold X has a finite dimensional finite skeleton. In particular, all homology groups are finitely generated (*i.e.* it consists of a free and a torsion part). Moreover, if X is closed (*i.e.* it has no boundary), connected and orientable, then by Poincaré duality, there is a canonical isomorphism $H_p(X, \mathbb{Z}) = H^{n-p}(X, \mathbb{Z})$. However by universal coefficient formulae

$$H^{n-p}(X, \mathbb{Z}) \simeq \operatorname{Hom}\left(H_{n-p}(X, \mathbb{Z}), \mathbb{Z}\right) \oplus \operatorname{Ext}\left(H_{n-p-1}(X, \mathbb{Z}), \mathbb{Z}\right).$$

Note that

$$\begin{aligned} \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) &= \mathbb{Z} \qquad & \operatorname{Hom}(\mathbb{Z}_p, \mathbb{Z}) &= 0 \\ \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) &= 0 \qquad & \operatorname{Ext}(\mathbb{Z}_p, \mathbb{Z}) &= \mathbb{Z}_p. \end{aligned}$$

These calculations imply that $H_p(X, \mathbb{Z})$ and $H_{n-p}(X, \mathbb{Z})$ have the same free part, and the torsion part of $H_p(X, \mathbb{Z})$ is the same as the torsion part of $H_{n-p-1}(X, \mathbb{Z})$. Now, for X a K3 surface, simply put $n = 4$, $p = 2$. Since $H_1(X, \mathbb{Z})$ is torsion-free, so is $H_2(X, \mathbb{Z})$.

So $H^2(X, \mathbb{Z})$ is a free $\mathbb{Z}$ module of rank 22. Moreover, there is a natural integral bilinear product on it, namely the cup-product,

$$\cup : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow H^4(X, \mathbb{Z}) \simeq H_0(X, \mathbb{Z}) \simeq \mathbb{Z}.$$

Thus, $H^2(X, \mathbb{Z})$ equipped with this product forms a lattice which we denote by L_X for short. Moreover, by Poincaré duality, this pairing is unimodular. By Milnor's classification of unimodular lattices (see [12],[13],[17],[20]), such a lattice is determined uniquely by its rank, signature and parity. By Hodge index theorem [2, §IV 2.13], we have $\operatorname{sign} L_X = (3, 19)$, and by Wu's formula $\alpha^2 \equiv \alpha \cdot c_1(\mathcal{K}_X) \equiv 0 \mod 2$, $\forall \alpha \in H^2(X, \mathbb{Z})$, *i.e.* the product is even. Now, it follows from Milnor's theorem that $L_X \simeq E_8^2 \oplus U^3$ where E_8 is the negative definite root lattice of rank 8, and U is the hyperbolic plane.

A surface E is called an *Enriques surface* if the geometric genus $p_g(E) = 0$, the irregularity $q(E) = 0$ and the square of the canonical bundle $\mathcal{K}_E^{\otimes 2} = \mathcal{O}_E$. Enriques surfaces are not simply-connected, as $\pi_1(E) \simeq \mathbb{Z}_2$. By similar calculations as above, $\chi(\mathcal{O}_E) = 1$, $\chi(E) = 12$, $b_0 = b_4 = 1$, $b_1 = b_3 = 0$ and $b_2 = 10$. Thus, $H^2(E,\mathbb{Z}) \simeq \mathbb{Z}^{10} \oplus \mathbb{Z}_2$, and by Milnor's theorem, its free part is isometric to $E_8 \oplus U =: \Theta$ which is called the *Enriques lattice.*

The relation between K3 and Enriques surfaces is given in

Theorem 1.1.1 (see [3, §VIII.17]) *The universal (unbranched) double cover of any Enriques surface is K3. Conversely, any K3 surface X with a fixed point free involution $\imath$ doubly-covers an Enriques surface, namely $X/\imath$.*

Assume that the K3 surface X covers an Enriques surface E. Then the fixed point free involution $\imath : X \to X$ induces an involution homomorphism $\imath^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$. This is an isomorphism between two integral lattices, so it can have only two eigenvalues ± 1 over $\mathbb{Z}$. The positive eigenspace (called the *invariant sublattice*) is denoted by L_X^+ and is isometric to $E_8(2) \oplus U(2) =: \Lambda^+$. The negative eigenspace (called the *anti-invariant sublattice*) is denoted by L_X^- and is isometric to $E_8(2) \oplus U(2) \oplus U =: \Lambda^-$ (see [14]) .

Since $p_g = 0$, by Hodge decomplosition we have $H^2(E,\mathbb{Z}) = \mathrm{Pic}(E)$, *i.e.* Enriques surfaces are algebraic. If we pullback (algebraic) cycles on E by p^*, where $p : X \to E$ is the double covering, we get algebraic cycles that are invariant under $\imath^*$, *i.e.* cycles in L_X^+. But by [14, Prop. 2.3] due to Mukai, it turns out that these are all the cycles in L_X^+. In particular, this means $NS(X)$ contains L_X^+. Although for a generic K3 surface covering an Enriques surface $NS(X) \cap L_X^- = \emptyset$, in the non-generic case this intersection is nonempty, *i.e.* there are algebraic cycles that are anti-invariant under $\imath^*$. To understand these cycles one needs to study the lattice $L_X^- \simeq \Lambda^-$ closely.

1.2 Periods of Enriques surfaces

Let X be any K3 surface. Denote the class of a nowhere vanishing holomorphic two form Ω by $\omega \in H^0(X, \Omega_X^2)$. By De Rham theorem and the Hodge decomposition we can regard ω as a point in $H^2(X, \mathbb{C})$. Since $p_g = 1$, this ω is unique up to a multiplicative constant.

Now, assume that X covers an Enriques surface E. Let $\imath : X \to X$ be the involution on X. Choose a marking (*i.e.* a fixed isometry) $\varphi : H^2(X, \mathbb{Z}) \to \Lambda$ such that $\varphi \circ \imath^* = \rho \circ \varphi$ where

$$\begin{aligned} \Lambda = E_8 \oplus E_8 \oplus U \oplus U \oplus U \quad &\to \quad \Lambda \\ \rho(e_1, e_2, u_1, u_2, u_3) \quad &= \quad (e_2, e_1, -u_2, -u_1, u_3). \end{aligned}$$

Such a marking always exists (see [7, 5.1], [2, §VIII 19.1]). Since X covers an Enriques surface, we must have $\imath^*(\omega) = -\omega$; because otherwise (*i.e.* $\imath^*(\omega) = \omega$) ω would induce a holomorphic 2-form on E, which is impossible since $p_g(E) = 0$. So $\omega \in (L_X^-)_{\mathbb{C}}$ and $\varphi(\omega) \in \Lambda_{\mathbb{C}}^-$. Therefore, to a given Enriques surface we can associate a point $\varphi(\omega) \in \Lambda_{\mathbb{C}}^-$. But since w is unique up to a multiplicative constant, this defines a line in Λ^- passing through the origin, and this further defines a point (ω) in $\mathbb{P}(\Lambda_{\mathbb{C}}^-)$. Moreover $\omega \in H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C}$ must also satisfy the Riemann relations $\omega \cdot \omega = 0$ and $\omega \cdot \overline{\omega} > 0$. So, at the end of this process we get a point in

$$\mathcal{D} = \{(\omega) \in \mathbb{P}(\Lambda_{\mathbb{C}}^-) : \omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0\}.$$

Note that this association essentially depends on the marking φ, but it is unique mod Γ, where

$$\Gamma = \mathrm{restr}_{\Lambda^-}\{g \in O(\Lambda) : g\rho = \rho g\}.$$

This defines the *period map*

$$\begin{aligned} \Phi : \text{Enriques Surfaces} \quad &\longrightarrow \quad \mathcal{D}/\Gamma \\ E \quad &\mapsto \quad (w) \mod \Gamma. \end{aligned}$$

Namikawa [14] showed that in fact $\Gamma = O(\Lambda^-)$. This period map was first studied by Horikawa [7],[8]. He proved that *(i)* Weak Torelli Theorem holds for Enriques surfaces, *i.e.* $\Phi(E) = \Phi(E')$ implies $E \simeq E'$.

(ii) The image of Φ is everything except the divisor $H := \bigcup_{l^2=-2} H_l$, where $H_l = \{(w) \in \mathbb{P}(\Lambda^-_{\mathbb{C}}) : w \cdot l = 0\}$. That is, Φ is surjective onto $\dfrac{\mathcal{D} - H}{\Gamma}$. This is called *the moduli space of Enriques surfaces.*

Chapter 2

Orbits in Λ^-

In the previous chapter we have seen that the lattice Λ^- and its automorphism group $O(\Lambda^-)$ appear in various geometrical contents related to the algebraic curves on K3 surfaces and to the moduli space of Enriques surfaces. However, Λ^- is a quite complicated lattice; its discriminant is equal to 1024 and its signature is $(2, 10)$. In general, $O(\Lambda^-)$ is also very hard to deal with; for instance, given any two vectors in Λ^- known to be equivalent $\mod O(\Lambda^-)$, it is virtually impossible to construct an isometry in $O(\Lambda^-)$ mapping one vector to the other.

In this chapter we will determine all orbits of the action of $O(\Lambda^-)$ on Λ^-. Our proof is inspired by a lattice-theoretical trick of Allcock [1] and a theorem of Wall [20, Theorem 4]. It will turn out that the orbit of a vector depends only on its norm, divisor and type. We will also count the number of orbits in the set of primitive $(2n)$-vectors. The chapter will close with a characterization of these orbits for primitive cohomology classes in L_X^- on any K3 surface X.

2.1 Definitions

By Milnor's classification theorem for indefinite unimodular lattices, any odd lattice with signature (s, t) is isomorphic to $I_{s,t} := \langle 1\rangle^s \oplus \langle -1\rangle^t$ and any even one to $II_{s,t} := (E_8(\pm 1))^{\frac{a-b}{8}} \oplus U^b$ where $a = \max\{s,t\}$ and $b = \min\{s,t\}$. In particular, signature of any even indefinite unimodular lattice is divisible by 8.

The type of an element $\omega \in L$ is defined to be *characteristic* if $\omega \cdot \eta \equiv \eta \cdot \eta \mod 2$ for all $\eta \in L$, and *ordinary* otherwise. According to a theorem of Van der Blij [13, §II 5.2] if a vector ω in a unimodular lattice L is characteristic, then $\omega \cdot \omega \equiv \mathrm{sign}(L) \mod 8$.

2.2 Allcock's trick

In [1], Allcock used a subtle lattice-theoretical trick to show that $O(B(2) \oplus U) \simeq O(I_{s,t})$, where B is any even indefinite unimodular lattice of signature $(s-1, t-1)$ (hence, B is isometric to $II_{s-1,t-1}$). Note that putting $B = E_8 \oplus U$ gives the isomorphism $O(\Lambda^-) \simeq O(I_{2,10})$. The latter isomorphism had also been discovered by Kondō by different methods in 1994 (see [10]).

Allcock's argument is as follows: Let A be any lattice isometric to $B(2) \oplus U$. Then $(\frac{1}{\sqrt{2}}A)^* \simeq B \oplus U(2)$. Now, notice that the primitive embeddings of a lattice L into a unimodular lattice is characterized by the nontrivial elements of its discriminant group L^*/L. In our case, for $L = B \oplus U(2)$ this group has $\mathrm{discr}\, L = 4$ elements, and it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that only the embedding corresponding to the element $(1,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ gives an embedding into an odd unimodular lattice, say $\hat{A}$. By Milnor's theorem $\hat{A} \simeq B \oplus I_{1,1}$ (here the uniqueness of embedding $L \hookrightarrow \hat{A}$ is crucial). Conversely, given any $\hat{A} \simeq B \oplus I_{1,1}$ then $A \simeq \sqrt{2}(\hat{A}^e)^*$ is isometric to $B(2) \oplus U$, where $\hat{A}^e$ is the maximal even sublattice of $\hat{A}$ (which is unique!). Considering A and $\hat{A}$ in $\mathbb{R}^n$ as euclidian lattices, it follows that any isometry of one of them preserves the other (after extending by linearity). This implies $O(B(2) \oplus U) \simeq O(B \oplus I_{1,1})$.

Here, we will give a coordinate-wise and constructive proof of Allcock's isomorphism. This method later will also help us to determine the orbits in Λ^-.

For any lattice L of rank n fix a basis and define

$$(\frac{1}{2})L = \{\frac{1}{2}L : \omega \in L \text{ and coordinates of } \omega \text{ have the same parity}\},$$

and extend the bilinear product on L linearly to $(\frac{1}{2})L$.

Now let $\operatorname{rank} B = (s-1)+(t-1) = s+t-2 =: r$

Lemma 2.2.1 *An element $\omega = (a_1, \ldots, a_r, m, n) \in B \oplus I_{1,1}$ is characteristic if and only if m and n are odd and a_i's are all even.*

Proof. Assume that ω is characteristic. Let $\eta = (0, \ldots, 0, 1, 0)$. Since $\omega \cdot \eta = m$ and $\eta \cdot \eta = 1$, it follows that m is odd. Similarly n is odd. Now, let $\eta = (c_1, \ldots, c_r, 0, 0)$ where $c = (c_1, \ldots, c_r)$ is obtained from the product $c^t = B^{-1}e_i$, where e_i is the column vector with 1 at i^{th} row and 0's elsewhere (here, we use the fact that B^{-1} actually exists since $\operatorname{discr} B = \pm 1$). Now, for $a = (a_1, \ldots, a_r)$ we have $c^t B a = a_i$. So, $w \cdot \eta = a_i$ and $\eta \cdot \eta \equiv 0 \mod 2$. Therefore a_i is even.

Converse is straightforward. □

Lemma 2.2.2 $O(B \oplus (\frac{1}{2})I_{1,1}) \simeq O(B \oplus I_{1,1})$.

Proof. $B \oplus I_{1,1}$ is a submodule of $B \oplus (\frac{1}{2})I_{1,1}$. After tensoring with $\mathbb{R}$, consider them as submodules in $\mathbb{R}^{r+2}$. Let $g \in O(B \oplus (\frac{1}{2})I_{1,1})$. It is easy to see that the matrix representation of g in the standard basis of $B \oplus (\frac{1}{2})I_{1,1}$ has integer entries. This matrix then defines an isometry of $B \oplus I_{1,1}$, thus an element in $O(B \oplus I_{1,1})$. This association is clearly injective. Conversely, take any isometry $g \in O(B \oplus I_{1,1})$. In order to extend g to all of $B \oplus (\frac{1}{2})I_{1,1}$ we have to define the image of an element ω of the form $w = (a_1, \ldots, a_r, m + \frac{1}{2}, n + \frac{1}{2})$. By previous lemma, 2ω is a characteristic element of $B \oplus I_{1,1}$, and so is $g(2w)$ because the type is an isometry invariant. Now, again by Lemma 2.2.1, $\frac{1}{2}g(2\omega)$ is in $B \oplus \frac{1}{2}I_{1,1}$. Now put $g(w) := \frac{1}{2}g(2\omega)$. This defines an isomorphism on $B \oplus (\frac{1}{2})I_{1,1}$ because its matrix is the same as the matrix of the initial g on $B \oplus I_{1,1}$, hence unimodular. □

Lemma 2.2.3 (Allcock's trick) *If B is an even indefinite unimodular lattice of signature $(s-1, t-1)$, then $O(B(2) \oplus U) \simeq O(B \oplus I_{1,1}) \simeq O(I_{s,t})$.*

Proof. Let $\{e_1, \ldots, e_r\}$ be the basis for $B(2)$, $\{u, v\}$ be the basis for U, and $\{x, y\}$ be the basis for $I_{1,1}$. We define a map

$$\begin{aligned} \phi : B(2) \oplus U &\longrightarrow B \oplus (\frac{1}{2}) I_{1,1} \\ (a_1, \ldots, a_r, b_1, b_2) &\mapsto (a_1, \ldots, a_r, \frac{b_1 + b_2}{2}, \frac{b_1 - b_2}{2}). \end{aligned}$$

Clearly, this is a $\mathbb{Z}$-module isomorphism with

$$\omega_1 \cdot \omega_2 = 2\ \phi(\omega_1) \cdot \phi(\omega_2)$$

for any $\omega_1, \omega_2 \in B(2) \oplus U$.

Such isomorphisms multiplying the form by a non-zero scalar are called *quasi-isometries*. Clearly, any two quasi-isometric ($\mathbb{Q}$-valued) lattices (not necessarily isometric) have isomorphic automorphism groups. Therefore $O(B(2) \oplus U) \simeq O(B \oplus (\frac{1}{2}) I_{1,1})$. On the other hand, by Milnor's theorem $O(B \oplus I_{1,1}) \simeq O(I_{s,t})$. Using the previous lemma we complete the proof. $\square$.

2.3 Main theorem

Using Allcock's trick and the quasi-isometry that we defined in the previous section, the problem of finding orbits in $B(2) \oplus U$ is 'reduced' to the same problem in $I_{s,t}$. However, the orbits in $I_{s,t}$ are already known [20]:

Theorem 2.3.1 (Wall) *If $s, t \geq 2$, then $O(I_{s,t})$ acts transitively on primitive vectors of given norm and type (i.e. characteristic or ordinary).*

Using this we deduce our

Main Theorem 2.3.2 *Let B be an even, indefinite, unimodular lattice. Consider the action of $O(B(2)\oplus U)$ on $B(2)\oplus U$. Then the set of primitive $(2n)$-vectors in $B(2)\oplus U$ consists of one orbit if n is odd, and two orbits if n is even.*

Again, letting $B = E_8 \oplus U$, the theorem applies to Λ^-.

We remind that some special cases of the theorem for Λ^- were proven by Namikawa (for $n = -1, -2$, [14, Theorems 2.13, 2.15]), by Allcock ($n = 0, -1$, [1]) and by Sterk ($n = 0, -2$, [18, 4.5]).

Proof of the main theorem. We will consider two cases:

Case 1: n is odd

Let $\omega = (a_1, \ldots, a_r, b_1, b_2)$ be a primitive vector in $B(2)\oplus U$. Then $\omega \cdot \omega = 4k + 2b_1b_2 = 2n$. Since n is odd, we have b_1 and b_2 both odd. So, $\phi(\omega) = (a_1, \ldots, a_r, \frac{b_1+b_2}{2}, \frac{b_1-b_2}{2})$ is an integral and primitive vector; moreover it is ordinary since $n \not\equiv 0 \mod 8$. Since $O(I_{s,t}) \simeq O(B \oplus I_{1,1})$, all such elements are equivalent mod $O(B \oplus I_{1,1})$. Since $B(2)\oplus U$ and $B \oplus I_{1,1}$ are quasi-isometric it turns out that all primitive $(2n)$-vectors in $B(2)\oplus U$ are transitive $\mod O(B(2)\oplus U)$. It remains to show the existence of such a vector. But clearly $w = (0, \ldots, 0, 1, n)$ is such a primitive $(2n)$-vector.

Case 2: n is even

Since $\omega \cdot \omega = 4k + 2b_1b_2 = 2n \equiv 0 \mod 4$, b_1 and b_2 cannot be both odd.

Case 2.1: Only one of b_1 and b_2 is even:

In this case, $\phi(\omega) = (a_1, \ldots, a_r, \frac{b_1+b_2}{2}, \frac{b_1-b_2}{2})$ has fractional coordinates. Instead, consider $2\phi(\omega)$, which is integral, primitive and, by lemma 1, characteristic. All such vectors are transitive by Wall's theorem. Therefore, all such ω's are transitive by similar arguments. Now, note that $w = (0, \ldots, 0, 1, n)$ in $B(2)\oplus U$ is such a primitive $(2n)$-vector.

Case 2.2: b_1 and b_2 are both even:

In this case, $\phi(\omega)$ is integral, primitive and ordinary. By similar arguments, ω's are again transitive under the action of $O(B(2)\oplus U)$. It remains to show the existence of such ω:

Since B is even, unimodular and indefinite, $B \simeq E_8(\pm 1)^i \oplus U^j$ where $j \geq 1$ and $i \geq 0$. For any $k \in \mathbb{Z}$, $(1,k) \in U(2)$ is a primitive vector of norm $4k$, so in particular $B(2)$ contains a primitive element ω of norm $4k$. Let $n = 2k$. Then $\omega := w' \oplus (0,0) \in B(2) \oplus U$ is the required primitive vector of norm $2n$.

Finally, note that the number of orbits of primitive $(2n)$-vectors is one if n is odd, and two if n is even. This completes the proof. □

Application Our theorem can be used to simplify the proofs of some theorems of Sterk [18]. In his paper, Sterk considers the action of a certain subgroup, which he calls Γ, of the group $O(\Lambda^-)$ [18, p.8]. He calculates that this action on the set of isotropic vectors has five orbits, each generated by primitive vectors e, e', $e' + f' + \omega$, $e' + 2f' + \alpha$, $2e + 2f + \alpha$. Here α and ω are some elements in $E_8(2)$ such that $\alpha^2 = -8$ and $\omega^2 = -4$. e, f are standard basis for U, and e', f' for $U(2)$ (see [18, 4.2.3]). He also claims that under the action of $O(\Lambda^-)$ the last four vectors are transitive, whereas the first vector lies in a different orbit (see [18, 4.5]). Using our theorem we can easily see this, because

$$\begin{aligned} e &= (0, \ldots, 1, 0) \\ e' &= (0, \ldots, 1, 0, 0, 0) \\ e' + f' + \omega &= (\omega_1, \ldots, \omega_8, 1, 1, 0, 0) \\ e' + 2f' + \alpha &= (\alpha_1, \ldots, \alpha_8, 1, 2, 0, 0) \\ 2e + 2f + \alpha &= (\alpha_2, \ldots, \alpha_8, 0, 0, 2, 2). \end{aligned}$$

Since all of the above vectors are primitive and isotropic (*i.e.* their norm is 0), we already know that there are exactly two orbits of such vectors. By the proof of the main theorem, the orbit of a primitive vector is determined by the parity of its last two coordinates. Indeed, all the vectors above except the vector e have the last two coordinate even, so they are transitive by Case 2.2. Note that e is not equivalent to them because it has one odd coordinate (Case 2.1).

Taking all $(2n)$-vectors into account including those that are not necessarily primitive we get the following:

Corollary 2.3.3 *Let* $\lambda(n) = \sum_{d^2|n,\ d>0} \frac{3+(-1)^{n/d^2}}{2}$. *Then the number of orbits of* $(2n)$*-vectors in* $B(2) \oplus U$ *is precisely* $\lambda(n)$.

Proof. Given two $(2n)$-vectors ω and ω'. Write $\omega = d\nu$ and $\omega' = d'\nu'$, where ν, ν' are primitive vectors, and d, d' are positive integers. Notice that this representation is unique. d is called the divisor of ω. Since divisor is an isometry invariant, it is clear that ω and ω' are not equivalent unless $d = d'$. In this case, ν and ν' are two primitive vectors of norm $2n/d^2$. Such vectors have two orbits if n/d^2 is even, and one orbit if it is odd, or shortly $\frac{3+(-1)^{n/d^2}}{2}$ orbits. In the case of ν and ν' are transitive under an isometry, the same isometry would map one of ω, ω' to the other. So, for fixed n, it suffices to sum these numbers over divisors d, *i.e.* over all positive integers d such that $d^2|n$. $\square$

Notice that in the proof of our theorem the orbit in which a $(2n)$-vector ω falls, depends only on the parity of the last two coordinates of ω in a fixed basis. However, for primitive cohomology classes in L_X^- on a K3 surface X we have also a basis-free characterization of those orbits:

A primitive $(2n)$-class $\omega \in L_X^-$ is defined to be of *even* parity if there is a primitive $(2n)$-vector $\omega' \in L_X^+$ such that $\omega + \omega' \in 2L_X$ (*cf.* [14, 2.16]).

Theorem 2.3.4 *Let* n *be an even integer. Let* ϕ *be the quasi-isometry defined in Lemma 2.2.3. A primitive* $(2n)$*-vector* $\omega \in L_X^-$ *is of even parity if and only if* $\phi(\alpha(\omega)))$ *has integral coordinates where* $\alpha : L_X^- \longrightarrow \Lambda^-$ *is any isometry.*

Proof. Since any self-isometry of $L_X^{\pm}$ extends to a self-isometry of L_X (see [14, 1.4]), without loss of generality we can fix a primitive embedding of Λ^- into Λ, and prove the statement for the image of this embedding. Therefore, we fix the following embedding

$$\begin{array}{rcl} \Lambda^- = E_8(2) \oplus U(2) \oplus U & \hookrightarrow & E_8^2 \oplus U^3 = \Lambda \\ (e, u, v) & \mapsto & (e, -e, u, -u, v) \end{array}$$

and identify the domain with its image in Λ. The orthogonal complement of the image is precisely the image of the primitive embedding

$$\begin{array}{rcl} \Lambda^+ = E_8(2) \oplus U(2) & \hookrightarrow & E_8^2 \oplus U^3 = \Lambda \\ (e,u) & \mapsto & (e,e,u,u,0). \end{array}$$

Moreover, the primitive $(2n)$-vectors $\omega \in \Lambda^-$ with integral images are transitive by Case 2.2. Therefore, it suffices to prove the statement for special vectors in Λ^-.

Let $\omega = (0,\dots,0,k,1,0,0) \in \Lambda^-$ be a primitive vector with $\omega^2 = 2n = 4k$, and identify ω with its image in Λ with coordinates $(0,\dots,0,k,1,-k,-1,0,0)$, by the above embedding. Notice that $\phi(\omega)$ has integral coordinates. Now choose $\omega' = (0,\dots,0,k,1) \in \Lambda^+$ which corresponds similarly to $(0,\dots,0,k,1,k,1,0,0) \in \Lambda$. Now it is clear that $\omega + \omega' \in 2\Lambda$.

On the other hand, the ϕ image of the vector $\omega = (0,\dots,0,2k,1) \in \Lambda^-$ has fractional coordinates, and for no vector ω' in Λ^+ can we have $\omega + \omega' \in 2\Lambda$, because the last coordinate of $\omega + \omega'$ is always 1.

This completes the proof. $\square$

Now, a primitive $(2n)$-vector in Λ^- with n even is called of *odd* parity if its ϕ-image is fractional (case 2.1) and of *even* parity if it is integral (case 2.2). Equivalently, ω is even if $\omega \cdot \eta \equiv 0 \mod 2$, $\forall \eta \in \Lambda^-$, and odd otherwise. The equivalence follows from the fact that *(i)* even vectors have the last two coordinates even, *(ii)* any product in $E_8(2) \oplus U(2)$ is even. We extend the definition of this parity to arbitrary vectors ω in Λ^- by considering $\overline{\omega} := \omega/d$ where d is the divisor of ω (see the proof of Corollary 2.3.3). Then we have:

Corollary 2.3.5 *If n is odd, $O(\Lambda^-)$ acts transitively on $(2n)$-vectors having the same divisor. If n is even, $O(\Lambda^-)$ acts transitively on $(2n)$-vectors having the same parity and divisor.*

Proof. Let n be odd. Then for two $(2n)$-vectors ω, ω' with the same divisor d the primitive vectors ω/d and ω'/d are equivalent modulo isometries by the main theorem. By linearity, this implies that ω and ω' are equivalent as well.

The idea is similar for n even. However, one has to take the parity of ω and ω' into account. $\square$

Chapter 3

Geometric applications

In this chapter we will present some applications of the main theorem in Chapter 2. Some of these results are new, and some of them are due to Allcock, Namikawa and Sterk, who had proven our theorem for Λ^- for special values of n $(= 0, -1, -2)$.

It turns out that our theorem is quite useful to understand the moduli space of Enriques surfaces. Recall that the moduli space of Enriques surfaces is defined as $\dfrac{\mathcal{D} - H}{\Gamma}$ via periods. Here $\Gamma = O(\Lambda^-)$; $\mathcal{D}$ is a hermitian symmetric bounded domain of type IV in $\mathbb{P}^{11}$ cut out by Riemann relations; H is the union of hyperplanes orthogonal to (-2)-vectors in D with respect to the product of Λ^-. It is known that this space is rational [10] and quasi-affine [4]. In a sense, our theorem gives a rule for equivalence of 'rational' points in the moduli space.

On the other hand, the theorem seems to be less useful for the problems related to the 'anti-invariant' curves on K3 surfaces. The reason is that not any isometry of Λ^- is induced by an isometry of a K3 surface. The global Torelli theorem for K3 surfaces asserts that only those isometries $L_X \to L_X$ preserving the 'Hodge structure of weight 2' are induced by an automorphism of the surface [2, §VIII]. In particular, any such isometry must preserve the period and the 'positive cone' which are already a big restriction. Despite this, we will give here a survey of results about curves on K3 surfaces obtained by Namikawa and Sterk who studied the isometry groups of certain sublattices of Λ^- and the Néron-Severi lattice.

The first immediate applications are due to Allcock. By the quasi-isometry ϕ that we defined in Chapter 2, the (-2)-vectors are in 1-1 correspondence with the (integral) (-1)-vectors in $(E_8 \oplus U) \oplus I_{1,1} \simeq I_{2,10}$. So, we have a 'simpler' representation of the moduli space of Enriques surfaces.

Theorem 3.0.6 (Allcock) *The period map establishes a bijection between the isomorphism classes of Enriques surfaces and points of* $\dfrac{\mathcal{D} - H'}{\Gamma'}$ *where* $\Gamma' = O(I_{2,10})$, $H' = \bigcup\limits_{l^2=-1} H'_l$, $H'_l = \{(\omega) \in D : \omega \cdot l = 0\}$.

The fact that the (-2)-vectors are $O(\Lambda^-)$-transitive was first proven by Namikawa [14, 2.13] using intricate analysis of Nikulin [15] on primitive embeddings of non-unimodular lattices. Later, Allcock [1] gave an elegant proof using his trick that we described in Chapter 2. From this fact it easily follows

Theorem 3.0.7 (Namikawa, Allcock) H/Γ *is and irreducible divisor of* $\mathcal{D}/\Gamma$.

Our theorem can be used in order to generalize the above theorem:

Theorem 3.0.8 *Let* $\mathcal{N}_n = \bigcup N_l$, $N_l = \{(\omega) \in \mathcal{D} : \omega \cdot l = 0\}$ *where the union is taken over all primitive* $(2n)$*-vectors if* n *is odd, or all primitive* $(2n)$*-vectors of the same parity if* n *is even. Then* $\mathcal{N}_n/\Gamma \subset \mathcal{D}/\Gamma$ *is an irreducible divisor.*

The above theorem was also stated in Namikawa [14, 6.4] for $n = -2$.

Another known application of our theorem is related to the Satake-Baily-Borel compactification of this moduli space, $\overline{\mathcal{D}/\Gamma}$. The boundary components of this compactification is defined in terms of isotropic sublattices of Λ^-. In particular, orbits of isotropic vectors in Λ^- correspond to the 0-dimensional boundary components of $\overline{\mathcal{D}/\Gamma}$. It was Sterk [18, 4.5] who first proved that there are two orbits of such vectors. Later, Allcock showed it by using the trick. Our proof is in fact different from theirs, but in any case it implies this fact. So, we have:

Theorem 3.0.9 (Sterk, Allcock) *There are two* 0*-dimensional boundary components of* $\overline{\mathcal{D}/\Gamma}$.

Another application is the following theorem, though we don't give a proof here, due to Allcock:

Theorem 3.0.10 (Allcock) *The universal cover of* $\mathcal{D}_0 = \mathcal{D} - H$ *is contractible, as is the universal orbifold cover of* $\mathcal{D}_0/\Gamma$.

Now, we give a survey of results on curves on K3 and Enriques surfaces: Namikawa studied 'algebraic' (-2)-classes in L_X, and deduced the following theorems [14, 6.2].

Theorem 3.0.11 (Namikawa) *Let* E *be an Enriques surface,* X *its universal double cover,* $\omega_X \in H^2(X, \mathbb{C})$ *the period of* X. *Then* E *has a smooth rational curve if and only if there is a* (-4)*-vector in* $L_X^- \cap NS(X)$ *of even parity.*

As a corollary he gets [14, 6.5]:

Theorem 3.0.12 (Namikawa) *On an Enriques surface* E *there are only finitely many smooth rational curves modulo automorphisms of* E.

He also proves a similar theorem for elliptic curves [14, 6.7]:

Theorem 3.0.13 (Namikawa) *On an Enriques surface* E *there are only finitely many smooth elliptic curves up to* $Aut(E)$ *and linear equivalence.*

We remark that the K3 analogue of these theorems was proven by Sterk [19, 0.1]

Theorem 3.0.14 (Sterk) *Let X be a K3 surface. Then*
(1) The group $Aut(X)$ is finitely generated.
(2) For every $d \geq 2$, the number of $Aut(X)$-orbits in the collection of complete linear systems which contain an irreducible curve of self-intersection d is finite.

A corollary of this theorem is

Theorem 3.0.15 (Sterk) *$Aut(X)$ is finite if and only if X contains finitely many smooth rational curves.*

Chapter 4

Problems for further research

The main theorem in Chapter 2 can be used to study the moduli space problems for Enriques surfaces. A possible application is as follows:

Nikulin [16] defined a *root invariant* for an Enriques surface E. This invariant is a pair consisting of a lattice K and an inner product space over the field $\mathbb{Z}_2$. By definition, K is generated by Δ^- where

$$\Delta^- = \{\omega \in L_X^- \cap NS(X) : \omega^2 = -4, \exists\, \omega' \in L_X^+ \cap NS(X), w'^2 = -4 \ s.t.\ \omega + \omega' \in 2L_X\}$$

where X is the double cover of E. The product in K is the product of L_X divided by 4. H is the kernel of the homomorphism

$$\begin{aligned} \xi : K/2K &\longrightarrow L_X^+ \cap NS(X)/2(L_X^+ \cap NS(X)) \\ \omega \mod 2 &\mapsto \omega' \mod 2. \end{aligned}$$

The following problem is suggested by Igor Dolgachev [5]: Stratify the moduli space of all Enriques surfaces in terms of moduli spaces of Enriques surfaces with fixed root invariant. That is, find a representation as

$$\mathcal{M}_E = \coprod_{\Delta} \mathcal{M}_E(\Delta)$$

where Δ stands for a root invariant (K, H).

For instance, let us find $\mathcal{M}_E(\Delta_0)$ where $\Delta_0 = (K, H)$ with $\operatorname{rank} K = 1$ (in this case clearly ξ is injective, and so $H = \{0\}$). These correspond to Enriques surfaces with a unique rational curve C. Then, on the K3 surface X, we have $\pi^*(C) = R + R'$ where R and R' are two disjoint rational curves on X. Then K is indeed generated by $R - R'$. Now, the period of such Enriques surfaces are in $C^\perp \cap \mathcal{D}$ where $C^\perp$ consists of vectors orthogonal two C. Since all such C's are equivalent modulo $O(\Lambda^-)$, so are the hyperplanes $C^\perp$, and we deduce that in fact all such Enriques surfaces lie on the same irreducible divisor in the moduli space.

By the way, we should point out that not any pair (K, H) is realized by an Enriques surface. K has rank at most 10, and it is a root system, *i.e.* direct some of A_k, D_k, E_k's. It is an interesting problem to find all such root invariants and classify them.

Nikulin in his paper [16] in 1984 gave a list of six root invariants of Enriques surfaces having a finite automorphism group. Later, in 1986, Kondō completed the list by adding a seventh root invariant non-isomorpic to the previous six and still realized by an Enriques surface with finite automorphism group (see [11]).

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